JOURNAL OF APPROXIMATION THEORY 53, 145–154 (1988)

Trigonometric Interpolation in Hölder Spaces

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Communicated by G. Meinardus

Received May 1, 1985

This note generalizes estimates in [8] for approximation of periodic functions by Fourier sums and interpolatory polynomials in Hölder spaces. In particular, we give explicit values for constants appearing in Hölder norm results.

1. INTRODUCTION

Let X be one of the usual spaces C or L^p $(1 \leqslant p < \infty)$ of 2π-periodic complex-valued functions. If $f \in C$, we write $\|f\|_\infty$ instead of $\|f\|_C$. For $0 \leqslant \alpha \leqslant 1$ and $m = 0, 1, 2, ...$, we denote by $X^{m,\alpha}$ the class of functions f which fulfil the following condition [2, Definition 1.5.5.]: There exists a 2π-periodic $(m-1)$-times absolutely continuous function ϕ with $\phi^{(m)} \in X$ ($\phi \in X$ in the case $m = 0$), $f = \phi$ in X and

$$\|\phi^{(m)}\|_{p,\alpha} := \sup_{h \neq 0} |h|^{-\alpha} \|\phi^{(m)}(\circ + h) - \phi^{(m)}(\circ)\|_p < \infty.$$

A norm in $X^{m,\alpha}$ is given by

$$\|f\|_{p,m,\alpha} := \sum_{k=0}^{m} \|\phi^{(k)}\|_p + \|\phi^{(m)}\|_{p,\alpha}. \tag{1}$$

We consider, for $f \in X^{m,\alpha}$, the nth Fourier sum

$$(S_n f)(x) = \frac{1}{\pi} \int_0^{2\pi} f(x-u)\, K_n(u)\, du$$

with the Dirichlet kernel

$$K_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx.$$

0021-9045/88 $3.00

Further, let $L_n f$ be the trigonometric interpolatory polynomial of degree n of a function $f \in C^{m,\alpha}$, based on the equidistant nodes $x_k = 2k\pi/(2n+1)$ $(k = 0, 1, ..., 2n)$:

$$(L_n f)(x) = \frac{2}{2n+1} \sum_{k=0}^{2n} f(x_k)\, K_n(x - x_k).$$

We use the theorem of Jackson on the order of approximation in the following form:

THEOREM [1, Chap. 5]. *For $n = 0, 1, ..., m = 0, 1, ..., 0 \leqslant \alpha \leqslant 1$ and $f \in X^{m,\alpha}$, we have*

$$E_n(f, X) \leqslant 3(n+1)^{-m-\alpha}\, \|\phi^{(m)}\|_{p,\alpha} \tag{2}$$

and

$$E_n(f, X) \leqslant \frac{\pi}{2}(n+1)^{-m}\, \|\phi^{(m)}\|_p. \tag{3}$$

2. Operator Norms and Approximation by Fourier Sums

The norm of the operators S_n and L_n is estimated as follows.

LEMMA 1. *For $n \geqslant 1$ we have*

$$\|S_n\|_{X \to X} \leqslant \begin{cases} \dfrac{4}{\pi^2} \ln n + c_n, & \text{if } 1 \leqslant p \leqslant \infty, \\ A_p, & \text{if } 1 < p < \infty \end{cases} \tag{4}$$

with

$$c_n = \begin{cases} 1,436, & \text{if } n = 1, \\ 1,362, & \text{if } n > 1, \end{cases}$$

and

$$A_p = \begin{cases} 4(p/(p-1))^{1/p} + 1, & \text{if } 1 < p < 2, \\ 1, & \text{if } p = 2, \\ 4p^{1-1/p} + 1, & \text{if } 2 < p < \infty. \end{cases}$$

Proof. To prove the first part of (4) we use the well-known convolution theorem [2, p. 10]

$$\|S_n f\|_p \leqslant \|f\|_p \, \|2K_n\|_1.$$

We get an estimation of $\|K_n\|_1$ from the representation of the remainder in [5]. In the case $1 < p < \infty$ the inequality (4) is an immediate consequence of

$$\|S_n f\|_p \leqslant A_p \, \|f\|_p \qquad \text{for} \quad f \in L^p \ (1 < p < \infty),$$

which is proved in [9, Chap. 7]. ∎

LEMMA 2. *We have for $n \geqslant 1$*

$$\|L_n\|_{C \to X} \leqslant \begin{cases} 1, & \text{if} \quad 1 \leqslant p \leqslant 2, \\ 3A_p, & \text{if} \quad 2 < p < \infty, \\ \dfrac{2}{\pi} \ln n + C_n, & \text{if} \quad 2 < p \leqslant \infty, \end{cases}$$

with

$$C_n = \begin{cases} 5/3, & \text{if} \quad n = 1, \\ 1,548, & \text{if} \quad n > 1. \end{cases}$$

Proof. The Parseval equation [9, Chap. 10.2]

$$\|L_n f\|_2^2 = \frac{1}{2n+1} \sum_{k=0}^{2n} |(L_n f)(x_k)|^2$$

yields

$$\sup_{\|f\|_\infty = 1} \|L_n f\|_p \leqslant \sup_{\|f\|_\infty = 1} \|L_n f\|_2 = 1, \qquad \text{if} \quad 1 \leqslant p \leqslant 2.$$

Generally for $1 < p < \infty$ the assertion follows from [9, Chap. 10]

$$\|L_n f\|_p \leqslant 3A_p \left\{ \frac{1}{2n+1} \sum_{k=0}^{2n} |(L_n f)(x_k)|^p \right\}^{1/p}.$$

If $X = C$, it is proved in [4]

$$\|L_n\|_{C \to C} = \frac{1}{2n+1} \left\{ 1 + 2 \sum_{k=0}^{n-1} \left(\sin \frac{2k+1}{4n+2} \pi \right)^{-1} \right\}. \tag{5}$$

Since

$$|(\sin x)^{-1} - x^{-1}| \leqslant 1 - 2/\pi \qquad \text{for} \quad |x| \leqslant \pi/2, x \neq 0$$

and

$$|(\sin x)^{-1} - x^{-1}| \leqslant \sqrt{2} - 4/\pi \qquad \text{for} \quad |x| \leqslant \pi/4, x \neq 0$$

by the monotonicity of the left side for $0 < x \leqslant \pi/2$, it follows that

$$\|L_n\|_{C \to C} \leqslant \frac{1}{2n+1} + \frac{n}{2n+1}\left(1 + \sqrt{2} - \frac{6}{\pi}\right) + \frac{2}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)\,2}{(2k+1)\,\pi}.$$

For $n \geqslant 20$ we estimate

$$\|L_n\|_{C \to C} \leqslant \frac{1}{39} + \frac{1}{2} + \frac{\sqrt{2}}{2} - \frac{3}{\pi} + \frac{4}{\pi} \sum_{k=0}^{19} \frac{1}{2k+1} + \frac{2}{\pi} \ln(2n-1) - \frac{2}{\pi} \ln 39$$

$$< 1,545 + \frac{2}{\pi} \ln n$$

as we stated. For $n = 1, 2, ..., 19$ we get the assertion by easy calculations in (5). ∎

Remark. With the same methods but sharper estimates we obtain $C_n < 1,5$ for $n \geqslant 4$ and $c_n < 1,3$ for $n \geqslant 7$.

THEOREM 1. *Supposing* $f \in X^{m,\alpha}$ *with* $0 \leqslant r \leqslant m$, $0 \leqslant \alpha, \beta \leqslant 1$, $r + \beta \leqslant m + \alpha$, $n \geqslant 1$, *we have*

$$\|f - S_n f\|_{p,r,\beta} \leqslant B(n,p)((n+1)^{\beta} + 1)(n+1)^{r-m-\alpha} \|\phi^{(m)}\|_{p,\alpha}$$

with

$$B(n,p) = \begin{cases} 14,616 + \dfrac{24}{\pi^2} \ln n, & \textit{if} \quad 1 \leqslant p \leqslant \infty, \\[2ex] 6A_p + 6, & \textit{if} \quad 1 < p < \infty. \end{cases}$$

Proof. Using $(S_n f)^{(k)} = S_n \phi^{(k)}$ $(0 \leqslant k \leqslant r)$, we get

$$\|\phi^{(k)} - (S_n \phi)^{(k)}\|_p \leqslant (1 + \|S_n\|_{X \to X})\, E_n(\phi^{(k)}, X). \tag{6}$$

With the notation

$$B_p = 1 + \|S_n\|_{X \to X}$$

it follows from (2) and (6) that

$$\|\phi^{(k)} - (S_n \phi)^{(k)}\|_p \leqslant 3B_p (n+1)^{k-m-\alpha} \|\phi^{(m)}\|_{p,\alpha}. \tag{7}$$

Hence we obtain by summation

$$\sum_{k=0}^{r} \|\phi^{(k)} - (S_n\phi)^{(k)}\|_p \leqslant 6B_p(n+1)^{r-m-\alpha} \|\phi^{(m)}\|_{p,\alpha}. \tag{8}$$

Now we define for brevity $G = \{h\colon |h| > 1/(n+1)\}$, $H = \{h\colon 0 < |h| \leqslant 1/(n+1)\}$, and

$$g_h(x) = \phi(x+h) - \phi(x). \tag{9}$$

Applying (7) we get for $h \in G$,

$$\begin{aligned}\sup_{h\in G} |h|^{-\beta} \|g_h^{(r)} - S_n g_h^{(r)}\|_p &\leqslant 2(n+1)^{\beta} \|\phi^{(r)} - S_n\phi^{(r)}\|_p \\ &\leqslant 6B_p(n+1)^{r-m+\beta-\alpha} \|\phi^{(m)}\|_{p,\alpha}.\end{aligned} \tag{10}$$

In the case $h \in H$ we estimate with (6)

$$\sup_{h\in H} |h|^{-\beta} \|g_h^{(r)} - S_n g_h^{(r)}\|_p \leqslant \sup_{h\in H} |h|^{-\beta} B_p E_n(g_h^{(r)}, X).$$

If $\beta \leqslant \alpha$ it follows easily from (3)

$$\begin{aligned}\sup_{h\in H} |h|^{-\beta} B_p E_n(g_h^{(r)}, X) &\leqslant \sup_{h\in H} |h|^{-\beta} B_p \frac{\pi}{2} (n+1)^{r-m} \|g_h^{(m)}\|_p \\ &\leqslant B_p \frac{\pi}{2} (n+1)^{r-m+\beta-\alpha} \|\phi^{(m)}\|_{p,\alpha}.\end{aligned} \tag{11}$$

Let now $\alpha < \beta$ which implies $r < m$. Applying Jackson's theorem we obtain

$$\begin{aligned}&\sup_{h\in H} |h|^{-\beta} B_p E_n(g_h^{(r)}, X) \\ &\quad\leqslant \sup_{h\in H} |h|^{-\beta} B_p\, 3(n+1)^{r-m+1} \sup_{\delta\in H} \|g_h^{(m-1)}(\circ+\delta) - g_h^{(m-1)}(\circ)\|_p \\ &\quad= 3(n+1)^{r-m+1} B_p \sup_{h\in H} \sup_{\delta\in H} |h|^{-\beta} \left\| \int_0^h g_\delta^{(m)}(\circ+u)\, du \right\|_p \\ &\quad\leqslant 3(n+1)^{r-m+1} B_p \sup_{h\in H} |h|^{1-\beta} \sup_{\delta\in H} \|g_\delta^{(m)}\|_p \\ &\quad\leqslant 3B_p(n+1)^{r-m+\beta-\alpha} \|\phi^{(m)}\|_{p,\alpha}.\end{aligned} \tag{12}$$

The proof is complete, if we summarize (8), (10)–(12). ∎

The nth Fourier sum of a function $f \in L^2$ is also the trigonometric polynomial of best approximation to f in L^2. Therefore we can sharpen the theorem in this case. Under the same conditions it holds that

$$\| f - S_n f \|_{2,r,\beta} \leqslant 6((n+1)^{\beta}+1)(n+1)^{r-m-\alpha} \|\phi^{(m)}\|_{2,\alpha}.$$

3. Interpolation

At first we estimate the difference between Fourier sum $S_n f$ and interpolatory polynomial $L_n f$ for functions $f \in C^{m,\alpha}$.

Theorem 2. *Supposing* $f \in C^{m,\alpha}$ *with* $0 \leqslant r \leqslant m$, $0 \leqslant \alpha, \beta \leqslant 1$, $r+\beta \leqslant m+\alpha$, $n \geqslant 1$, *we have*

$$\|S_n f - L_n f\|_{p,r,\beta} \leqslant C(n,p)((n+1)^{\beta}+1)(n+1)^{r-m-\alpha} \|f^{(m)}\|_{C,\alpha}$$

with

$$C(n,p) = \begin{cases} 12, & \textit{if } \ 1 \leqslant p \leqslant 2, \\ 24 A_p, & \textit{if } \ 2 < p < \infty, \\ 18,616 + \left(\dfrac{12}{\pi} + \dfrac{24}{\pi^2}\right) \ln n, & \textit{if } \ 2 < p \leqslant \infty. \end{cases}$$

Proof. Applying the inequality of Bernstein, we get

$$\begin{aligned} \|(S_n f)^{(k)} - (L_n f)^{(k)}\|_p &\leqslant n^k \|S_n f - L_n f\|_p \\ &\leqslant n^k(\|b_n - S_n f\|_p + \|b_n - L_n f\|_p), \end{aligned}$$

where b_n is the trigonometric polynomial of best approximation to f in C. With the help of (2) it follows that

$$\begin{aligned} \|(S_n f)^{(k)} - (L_n f)^{(k)}\|_p &\leqslant n^k E_n(f, C)(\|S_n\|_{C \to X} + \|L_n\|_{C \to X}) \\ &\leqslant 3(n+1)^{k-m-\alpha} \|f^{(m)}\|_{C,\alpha}(\|S_n\|_{C \to X} + \|L_n\|_{C \to X}). \end{aligned} \tag{13}$$

Further, we get

$$\sup_{h \neq 0} |h|^{-\beta} \|(S_n g_h)^{(r)} - (L_n g_h)^{(r)}\|_p \leqslant \sup_{h \neq 0} |h|^{-\beta} n^r \|S_n g_h - L_n g_h\|_p$$

with g_h defined in (9). Then we have

$$\sup_{h\in G} n^r\,|h|^{-\beta}\,\|S_n g_h - L_n g_h\|_p \leqslant \sup_{h\in G} 2n^r\,|h|^{-\beta}\,\|S_n f - L_n f\|_p$$

$$= 2n^r(n+1)^\beta\,\|S_n f - L_n f\|_p \qquad (14)$$

and

$$\sup_{h\in H} n^r\,|h|^{-\beta}\,\|S_n g_h - L_n g_h\|_p \leqslant \sup_{h\in H} n^r\,|h|^{-\beta}\,E_n(g_h, C)$$

$$\times\left(\|L_n\|_{C\to X} + \|S_n\|_{C\to X}\right). \qquad (15)$$

Now we must distinguish the cases $\beta\leqslant\alpha$ and $\alpha<\beta$ with $r<m$. We will consider only the first one, then the second case can be handled analogously as in (12).

Denoting the right side of (15) with A, we get for $\beta<\alpha$

$$A \leqslant \sup_{h\in H} \frac{\pi}{2}\, n^r(n+1)^{-m}\,|h|^{-\beta}\,\|g_h^{(m)}\|_C\left(\|S_n\|_{C\to X} + \|L_n\|_{C\to X}\right)$$

$$\leqslant \frac{\pi}{2}\, n^r(n+1)^{-m+\beta-\alpha}\,\|f^{(m)}\|_{C,\alpha}\left(\|S_n\|_{C\to X} + \|L_n\|_{C\to X}\right). \qquad (16)$$

Now it is enough to collect (13) for $0\leqslant k\leqslant r$, (14)–(16), to use $\|\circ\|_{p,r,\beta} \leqslant \|\circ\|_{2,r,\beta}$ for $1\leqslant p<2$ and we get the desired result. ▮

The following main theorem is now a simple consequence of the estimates for $\|f - S_n f\|_{p,r,\beta}$ and $\|S_n f - L_n f\|_{p,r,\beta}$.

THEOREM 3. *Supposing $f\in C^{m,\alpha}$ with $0\leqslant r\leqslant m$, $0\leqslant\alpha,\beta\leqslant 1$, $r+\beta\leqslant m+\alpha$, $n\geqslant 1$, we have*

$$\|f - L_n f\|_{p,r,\beta} \leqslant D(n,p)(n+1)^{r-m+\beta-\alpha}\,\|f^{(m)}\|_{C,\alpha}$$

with

$$D(n,p) \leqslant \begin{cases} 36, & \textit{if}\ \ 1\leqslant p\leqslant 2,\\ 60A_p + 12, & \textit{if}\ \ 2<p<\infty,\\ 66,464 + 17,367\ln n, & \textit{if}\ \ 2<p\leqslant\infty. \end{cases}$$

4. Corollaries

Let us give some refinements of this estimate:

1. If $n \geqslant 2$ we can improve the estimates for

$$\sum_{k=0}^{r} (n+1)^k$$

and for c_n and C_n introduced in Lemmas 1 and 2. That is why it holds for $n \geqslant 2$ that

$$D(n,p) \leqslant \begin{cases} 31{,}5, & \text{if } 1 \leqslant p \leqslant 2, \\ 52{,}5 A_p + 10{,}5, & \text{if } 2 < p < \infty, \\ 55{,}356 + 15{,}196 \ln n, & \text{if } 2 < p \leqslant \infty \end{cases}$$

and for $n \geqslant 7$ that

$$D(n,p) \leqslant \begin{cases} 28{,}286, & \text{if } 1 \leqslant p \leqslant 2, \\ 47{,}143 A_p + 9{,}429, & \text{if } 2 < p < \infty, \\ 48{,}086 + 13{,}645 \ln n, & \text{if } 2 < p \leqslant \infty. \end{cases}$$

Analogous improvements are possible for $B(n,p)$ in Theorem 1. The estimates are true for general n, p, r, β, m, α. It is expected that for special values the constants are essentially smaller.

2. Since $A_p \gg \ln n$ for small n, it is better to take the constant with the ln term at least for $n < 5 \cdot 10^8$ in Theorem 3 and at least for $n < 3.9 \cdot 10^5$ in Theorem 1.

3. It is easy to verify in the proofs that we can replace $\|\phi^{(m)}\|_{p,\alpha}$ in the right sides of Theorems 1–3 by

$$\sup_{h \in H} |h|^{-\alpha} \|\phi^{(m)}(\circ + h) - \phi^{(m)}(\circ)\|_p.$$

4. In [7] S. Prössdorf studied a closed subspace of $C^{0,\alpha}$. It is naturally to generalize this to the subspace $\tilde{X}^{m,\alpha} \subseteq X^{m,\alpha}$ $(0 \leqslant \alpha < 1)$:

$$\tilde{X}^{m,\alpha} = \left\{ f \in X^{m,\alpha} \colon \lim_{h \to 0} |h|^{-\alpha} \|\phi^{(m)}(\circ + h) - \phi^{(m)}(\circ)\|_p = 0 \right\}$$

with norm defined in (1). Then $\tilde{X}^{m,0} = X^{m,0}$.

Therefore we state a corollary.

COROLLARY. *Suppose* $0 \leqslant r \leqslant m$, $0 \leqslant \beta \leqslant 1$, $0 \leqslant \alpha < 1$ *and* $r+\beta \leqslant m+\alpha$. *Then it holds for* $f \in \tilde{X}^{m,\alpha}$

$$\|f - S_n f\|_{p,r,\beta} = \begin{cases} o(n^{r-m+\beta-\alpha} \ln n), & \text{if } \; p=1 \text{ or } p=\infty \\ o(n^{r-m+\beta-\alpha}), & \text{if } \; 1<p<\infty \end{cases} \qquad (n \to \infty)$$

and for $f \in \tilde{C}^{m,\alpha}$

$$\|f - L_n f\|_{p,r,\beta} = \begin{cases} o(n^{r-m+\beta-\alpha} \ln n), & \text{if } \; p=\infty, \\ o(n^{r-m+\beta-\alpha}), & \text{if } \; 1 \leqslant p<\infty, \quad (n \to \infty). \end{cases}$$

Remark. In [6] R. Haverkamp proved for $k \geqslant 1$ and $f \in C^{k,0}$,

$$\|f^{(k)} - (L_n f)^{(k)}\|_C \leqslant \left(1 + C_k \left(2 + \frac{2}{\pi} \ln n\right)\right) \mathring{E}_n(f^{(k)}, C) \tag{17}$$

with $C_k = ((\pi/2)^k(\pi+2) - 2\pi)/(\pi-2)$ and $\mathring{E}_n$ means that the infimum is taken over all trigonometric polynomials p_n of degree $\leqslant n$ with $S_0 p_n = 0$. From (6) and (13) we have for $f \in C^{k,0}$

$$\begin{aligned} \|f^{(k)} - (L_n f)^{(k)}\|_p \leqslant\, & (1 + \|S_n\|_{C \to X}) E_n(f^{(k)}, C) \\ & + n^k(\|S_n\|_{C \to X} + \|L_n\|_{C \to X}) E_n(f, C). \end{aligned} \tag{18}$$

With the inequality

$$E_n(f, C) \leqslant \mathring{E}_n(f, C) \leqslant \frac{\pi}{2n+2} \mathring{E}_n(f', C) \leqslant \left(\frac{\pi}{2n+2}\right)^k \mathring{E}_n(f^{(k)}, C)$$

which follows from [3, Chap. 43], we get

$$\|f^{(k)} - (L_n f)^{(k)}\|_p \leqslant D(k, n, p)\, \mathring{E}_n(f^{(k)}, C) \tag{19}$$

with

$$D(k, n, p) = 1 + \|S_n\|_{C \to X} + \left(\frac{\pi}{2}\right)^k (\|S_n\|_{C \to X} + \|L_n\|_{C \to X}).$$

Using Lemma 1 and 2 we obtain estimates for $D(k, n, p)$. If $k \geqslant 2$, $p = \infty$, they are better than (17). But, if we want to apply Jackson's theorem, it is useful to do this in (18) instead of (19), since $D(k, n, p) > 2(\pi/2)^k$. Such a result, where the constant is independent of k is contained in Theorem 3.

ACKNOWLEDGMENTS

I thank Dr. M. Tasche for the many helpful discussions and suggestions during the preparation of this manuscript.

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